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SCHOLARSHIP

Mana Tohu Mātauranga o Aotearoa
New Zealand Qualifications Authority

Scholarship 2025
Calculus

Time allowed: Three hours
Total score: 32

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

You should answer ALL the questions in this booklet.

Pull out Formulae and Tables Booklet S–CALCF from the centre of this booklet.

Show ALL working. Correct answers only will not be sufficient

If you need more room for any answer, use the extra space provided at the back of this booklet.

Check that this booklet has pages 2–24 in the correct order and that none of these pages is blank.

Do not write in any cross-hatched area (). This area will be cut off when the booklet is marked.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

QUESTION ONE

(a) Find all the real solutions of the following equation:

$\sqrt[3]{125^{x^3-5x^2+11x-3}} = \sqrt[4]{5^{4x^2+12}}$

$\left(5^{3(x^3-5x^2+11x-3)}\right)^{\frac{1}{3}} = 5^{x^3-5x^2+11x-3} = 5^{\frac{1}{4}(4x^2+12)} = 5^{x^2+3}$

$\Rightarrow x^3-5x^2+11x-3 = x^2+3 \Rightarrow x^3-6x^2+11x-6=0$

$\therefore x = 3, 2, 1$

None of these produce $5^{4x^2+12} < 0$, so all are valid solutions

(b) Find all the complex solutions of the following equation:

$z = 2 - \frac{2+i}{iz} \Leftrightarrow z^2 i = 2iz - 2 - i$

Let $z = x+yi$, hence $(x+yi)^2 i = 2i(x+yi) - 2 - i$

$(x^2 + 2xyi - y^2)i = 2ix - 2y - 2 - i$

$(x^2-y^2)i - 2xy = (2x-1)i - (2y+2)$

$\Rightarrow \begin{cases} x^2-y^2 = 2x-1 \\ 2xy = 2y+2 \Rightarrow 2 = 2xy-2y \Rightarrow 1 = y(x-1) \Rightarrow \cancel{x = \frac{1}{y}+1 = \frac{1+y}{y}} \end{cases}$ $y = \frac{1}{x-1}$

Hence $x^2 - \frac{1}{(x-1)^2} = \frac{x^2(x-1)^2-1}{(x-1)^2} = \frac{x^2(x^2-2x+1)-1}{(x-1)^2} = 2x-1 \Rightarrow x^2(x^2-2x+1)-1 = (2x-1)(x-1)^2$

$x^4-2x^3+x^2-1 = 2x^3-4x^2+2x-x^2+2x-1$

$\Rightarrow x^4-4x^3+6x^2-4x=0 \quad \therefore \quad x = \cancel{2.316}\ 2,\ y = \cancel{0.760}\ 1$

$x = \cancel{0.316}\ 0,\ y = \cancel{-0.760}\ -1$

$\therefore z_1 = 2+i, \quad z_2 = -i$

(c) How many different complex solutions does the equation $z^{2025} = \overline{z}$ have?

Justify your answer.

Note: as real numbers and purely imaginary numbers are both subsets of the complex numbers, you should consider them also.

$\cancel{z^3 - \overline{z} \Rightarrow}$ Let $z = x+yi$, $\frac{1}{\overline{z}} = \frac{1}{x-yi} \times \frac{x+yi}{x+yi} = \frac{x+yi}{x^2+y^2}$

$$\Rightarrow \frac{(x+yi)^{2025}(x+yi)}{x^2+y^2} = 1$$

$$\frac{(x+yi)^{2026}}{x^2+y^2} = 1$$

Let $x+yi = M\operatorname{cis}(\theta + 2n\pi)$ $n \in \mathbb{Z}$
where $M = |z|$ and $\theta = \arg(z)$

Hence by de Moivre's rule:

$$\frac{M^{2026}}{x^2+y^2} \operatorname{cis}\left(\frac{\theta}{2026} + \frac{n\pi}{1013}\right) = 1 \operatorname{cis}(0°)$$

Here $\frac{\theta}{2026} \leq \frac{n\pi}{1013} \leq 2\pi \Rightarrow \frac{n}{1013} \leq 2$ $\therefore n \leq 2026$

$$\frac{z^{2026}}{x^2+y^2} = 1$$

This is a complex polynomial of order 2026, hence the # of solutions = $\boxed{2026}$ by the fundamental theorem of algebra.

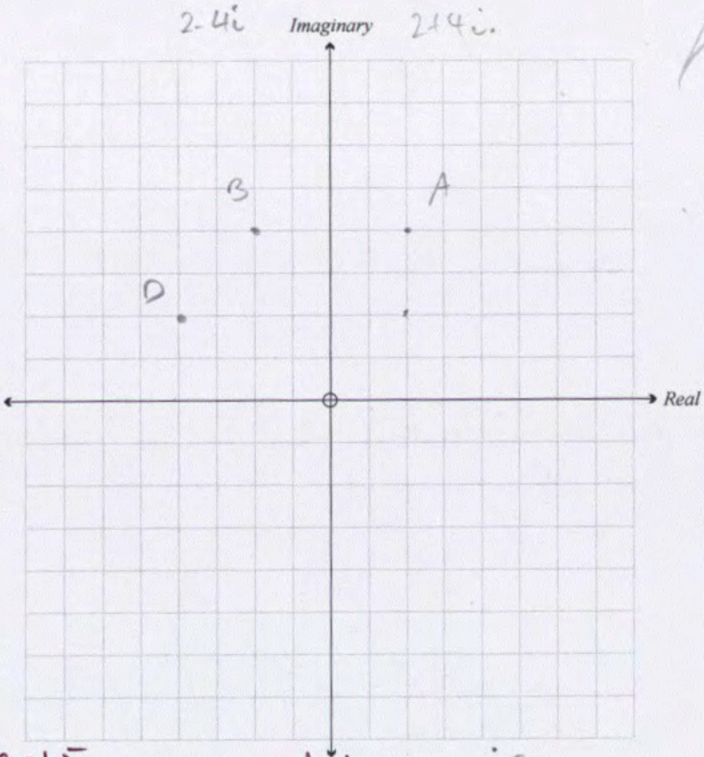
(d) Let $z = r\operatorname{cis}\theta$ be a complex number with $r > 1$ and $0 < \theta < \frac{\pi}{2}$.

Let the point A represent z in the Argand diagram and the points B, C, and D represent the complex numbers $\overline{z}, \frac{1}{z}$, and iz respectively.

B C D

If $\angle ABC = 60°$ and $BC = 1.5$, find the area of the quadrilateral ABCD.

Use the Argand diagram below to support your working.



60
45
4, 6

BC = 1.5

Consider ~~$\hat{ABC} = 60°$~~ $\Rightarrow$ the ~~complex sum~~ modulus of $C - B = 1.5$

let $z = x + yi$ $\Rightarrow$ $|C - B| = \left|\frac{1}{x+yi} - x + yi\right|$

~~hence~~ $\frac{1}{x+yi} = \frac{1}{x+yi} \times \frac{x-yi}{x-yi} = \frac{x-yi}{x^2+y^2}$ so

QUESTION TWO

(a) A function is defined parametrically by the equations:

$x = 2t + t^2$

$y = p(t)$

where:

- $p(t) = at^3 + bt^2 + ct + d$
- $p(1) = 2.5$ $\quad 3at^2+2bt+c$
- $p'(1) = 6$ $\quad 6at+2b$

Given that $\frac{d^2y}{dx^2} = \frac{3}{4(1+t)}$, find the value of $p(2)$.

① $p(1) = 2.5 = a+b+c+d$

② $p'(1) \overset{=6}{=} 3a + 2b + c \implies 3.5 = 2a + b - d \implies d = 2a+b-3.5$

$\frac{dx}{dt} = 2+2t$, $\frac{dy}{dt} = p'(t) \implies \frac{dy}{dx} = \frac{p'(t)}{2+2t}$

$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{p'(t)}{2+2t}\right) \cdot \frac{1}{2+2t} = \frac{p''(t)(2+2t) - 2p'(t)}{(2+2t)^3} = \frac{3}{4(1+t)}$

~~$p''(t)(2+2t) - 2p'(t) = (6at+2b)(2+2t) - (3at^2+2bt+c) = \frac{3}{4}(2+2t)^2$~~

~~LHS $= 12at+12at^2+4b+4bt-3at^2-2bt-c = 9at^2+(12a+2b)t+(4b-c)$~~

~~RHS $= \frac{3}{4}(4t^2+8t+4) = 3t^2+6t+3$~~

~~$\longrightarrow (9a-3)t^2+(12a+2b-6)t+(4b-c-3)=0$~~

$\implies p''(t)(2+2t) - 2p'(t) = \frac{3}{4}(2+2t)^2$

$(6at+2b)(2+2t) - 2(3at^2+2bt+c) = \frac{3}{4}(4+8t+4t^2)$

$12at+12at^2+4b+4bt-6at^2-4bt-2c = 3+6t+3t^2$

~~$6at^3+(12a-4b)t^2+(12a+4b-2c)t+4b = 3t^2+6t+3$~~

~~$6at + (4b-2c) = 3t^2+6t+3$~~

$6at^2+12at+4b-2c = 3t^2+6t+3$

$\therefore a = \frac{1}{2}$ and $4b-2c = 3$ ③

With ②: $6 = \frac{3}{2} + 2b + c \Rightarrow 2b + c = \frac{9}{2} \Rightarrow 4b + 2c = 9$

and ③: $4b - 2c = 3$

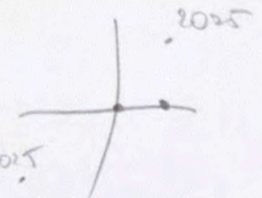
$\Rightarrow 8b = 12 \quad \therefore b = \frac{3}{2}, \; c = \frac{3}{2}$

Hence ①: $2.5 = 0.5 + 1.5 + 1.5 + d \quad \therefore d = -1$

$\therefore p(t) = \frac{1}{2}t^3 + \cancel{} \frac{3}{2}t^2 + \frac{3}{2}t - 1$

$\therefore p''(2) = \boxed{12}$

~~1~~ 2 3 4 5 6 7



(b) A function f is said to be **odd** if $f(-x) = -f(x)$ for all x in its domain.
Examples of odd functions include $f(x) = x^3$ and $f(x) = \sin(x)$.

$f(1-0) = f(1)$

Consider an **odd** function f that satisfies the condition $f(1-x) = f(1+x)$ for all real numbers.

Given that $f(1) = 2025$, find the value of $f(1) + f(2) + \ldots + f(2025)$.

~~$f(0) = 2025 = f(1-1) = f(1+1)$~~

~~Here $f(0) = f(2) = -f(-2) = -f(-1) = 2025$~~

$f(1) = -f(-1) = 2025$

~~$f(0) = f(2) = -f(-2)$~~ $f(0) = f(2) = -f(-2)$

$f(-1) = f(3) = -f(-3) = -2025$

~~$f(-2) = f(4) =$~~ $f(-2) = f(4) = -f(-4)$

$f(-3) = f(5) = -f(-5) = 2025$

Here $f(1) + f(3) + f(5) = 2025 - 2025 + 2025$

$\Rightarrow$ Sum of all odd terms $= 2025$

If function is odd, then $f(0) = 0$

Hence all even terms sum to 0

$\therefore f(1) + f(2) + f(3) \ldots + f(2025)$

$= 2025$

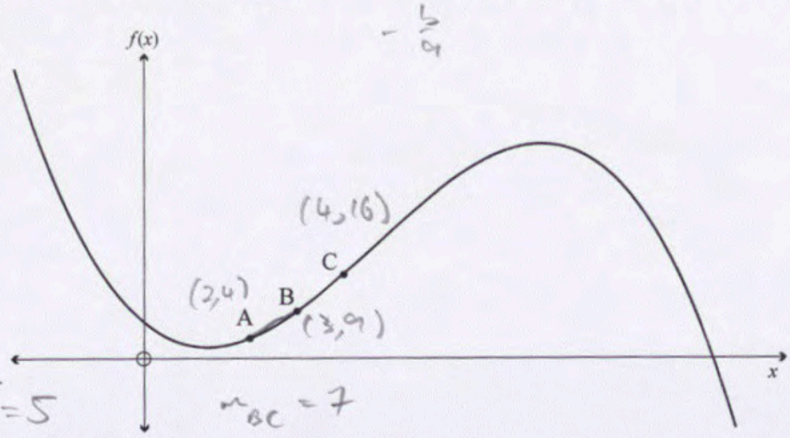
1 3 5 7 9 11

$f(1-x) = f(1+x) = -f(x-1) = -f(-x-1)$

(c) Consider the cubic function $f(x) = ax^3 + bx^2 + cx + d$.

The points A(2,4), B(3,9), and C(4,16) all lie on the graph of the cubic, as shown.

$-\frac{b}{a}$



$m_{AB} = \frac{5}{1} = 5$ $\quad m_{BC} = 7$

$m_{AC} = \frac{12}{2} = 6$

Suppose that the lines AB, AC, and BC intersect the cubic again at points D, E, and F, respectively, and that the sum of the x-coordinates of D, E, and F is 25.

Find the y-intercept of the cubic.

Hint: consider the function $g(x) = f(x) - x^2$ in factorised form.

A $\quad 4 = 8a + 4b + 2c + d$	$AB: y - 4 = 5(x-2) \Rightarrow y = 5x - 6$
B $\quad 9 = 27a + 9b + 3c + d$	$AC: y - 4 = 6(x-2) \Rightarrow y = 6x - 8$
C $\quad 16 = 64a + 16b + 4c + d$	$BC: y - 9 = 7(x-3) \Rightarrow y = 7x - 12$

1: $5x - 6 = ax^3 + bx^2 + cx + d$ (AB)

2: $6x - 8 =$ " " (AC)

3: $7x - 12 =$ " " (BC)

Sum of roots of 1: $\frac{b-5}{a}$

" " 2: $\frac{b-6}{a}$

" " 3: $\frac{b-7}{a}$

$\Rightarrow$ Sum of sum of roots $= \frac{3b-18}{a} = 25 + 2(2+3+4)$

$\frac{3b-18}{a} = 43$

$3b = 43a + 18$

(A) $4 = 8a$

(B) $9 =$

(C) $16 =$

$\frac{1-\sin x}{\cos x} \cdot \frac{1+\sin x}{1+\sin x} = \frac{1-\sin^2 x}{\cos x(1+\sin x)} =$

QUESTION THREE

(a) Prove that $\frac{1}{1+\sin x} = \sec x(\sec x - \tan x)$.

Hence, or otherwise, find the exact value of:

$$\int_0^{\frac{2\pi}{3}} \frac{1}{1+\sin x}\,dx$$ 1.2679

$RHS = \frac{1}{\cos x}\left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right) = \frac{1}{\cos x}\left(\frac{\cos x - \cos x \sin x}{\cancel{\cos x}\cos x}\right) = \frac{1-\sin x}{\cos^2 x} \times \frac{1+\sin x}{1+\sin x}$

$= \frac{1-\sin^2 x}{\cos^2 x(1+\sin x)} = \frac{\cos^2 x}{\cos^2 x(1+\sin x)} = \frac{1}{1+\sin x} = LHS$

$\Rightarrow \int_0^{2\pi/3} \frac{1}{1+\sin x}\,dx = \int_0^{2\pi/3} (\sec^2 x - \sec x \tan x)\,dx = [\tan x - \sec(x)]_0^{2\pi/3}$

$= [(-\sqrt{3}+2) - (0-1)] = \boxed{3-\sqrt{3}}$

(b) (i) It can be shown that $\tan\left(\frac{\pi}{8}\right) = \sqrt{2}-1$. $\tan\left(\frac{3\pi}{8}\right) = \sqrt{2}+1$

Find a similar expression for $\tan\left(\frac{3\pi}{8}\right)$.

Show all working.

$\tan\left(\frac{3\pi}{8}\right) = \tan\left(\frac{2\pi}{8}+\frac{\pi}{8}\right) = \frac{\tan\left(\frac{2\pi}{8}\right)+\tan\left(\frac{\pi}{8}\right)}{1-\tan\left(\frac{2\pi}{8}\right)\tan\left(\frac{\pi}{8}\right)}$

$\tan\left(\frac{2\pi}{8}\right) = \frac{\tan\left(\frac{\pi}{8}\right)+\tan\left(\frac{\pi}{8}\right)}{1-\tan^2\left(\frac{\pi}{8}\right)} = \frac{2(\sqrt{2}-1)}{1-(\sqrt{2}-1)^2} = \frac{2(\sqrt{2}-1)}{1-(3-2\sqrt{2})} = \frac{2(\sqrt{2}-1)}{-2+2\sqrt{2}}$

$= \frac{\sqrt{2}-1}{\sqrt{2}-1} = 1$

$\Rightarrow \tan\left(\frac{3\pi}{8}\right) = \frac{1+\tan\left(\frac{\pi}{8}\right)}{1-\tan\left(\frac{\pi}{8}\right)} = \frac{\sqrt{2}}{2-\sqrt{2}} \times \frac{2+\sqrt{2}}{2+\sqrt{2}} = \frac{2\sqrt{2}+2}{4-2}$

$\boxed{= \sqrt{2}+1}$

(ii) Hence, or otherwise, find all real solutions of the following system of equations in exact form:

$$\tan\left(\frac{x}{2}\right)\tan y = \sqrt{2} - 1 \quad (1)$$

$$\tan\left(\frac{x}{2} + y\right) = 1 + \sqrt{2} \quad (2)$$

$$(2):\ \tan\left(\tfrac{x}{2}+y\right) = \frac{\tan(\frac{x}{2}) + \tan(y)}{1 - \tan(\frac{x}{2})\tan(y)} = \frac{\tan(\frac{x}{2}) + \tan(y)}{2-\sqrt{2}} = 1+\sqrt{2}$$

$$\Rightarrow \tan\left(\tfrac{x}{2}\right) + \tan(y) = 2 + 2\sqrt{2} - \sqrt{2} - 2 = \sqrt{2}$$

$\tan(\frac{x}{2}+y) = 1+\sqrt{2} = \tan(\frac{3\pi}{8})$ by previous result

$$\Rightarrow \frac{x}{2} + y = \frac{3\pi}{8} \Leftrightarrow y = \frac{3\pi}{8} - \frac{x}{2}$$

$$\Rightarrow \tan\left(\tfrac{x}{2}\right) + \tan\left(\tfrac{3\pi}{8} - \tfrac{x}{2}\right) = \sqrt{2}$$

$$\tan\left(\tfrac{3\pi}{8} - \tfrac{x}{2}\right) = \frac{\tan(\frac{3\pi}{8}) - \tan(\frac{x}{2})}{1 + \tan(\frac{3\pi}{8})\tan(\frac{x}{2})} = \frac{\sqrt{2}+1-u}{1+u\sqrt{2}+u} \quad \text{where } u = \tan(\tfrac{x}{2})$$

$$\text{Hence } u + \frac{(\sqrt{2}+1)-u}{1+u(\sqrt{2}+1)} = \frac{u + u^2(\sqrt{2}+1) - u + (\sqrt{2}+1)}{1+u(\sqrt{2}+1)} = \sqrt{2}$$

$$\cancel{(\sqrt{2}+1)u^2 = \sqrt{2} + 2(2+\sqrt{2})u \Rightarrow (\sqrt{2}+1)u^2 - 2(2+\sqrt{2})u - \sqrt{2} = 0}$$

~~By GDC~~

$$\cancel{\text{Hence } u = \frac{4(2+\sqrt{2}) - \sqrt{4(4+4\sqrt{2}+2) + 4(2+\sqrt{2})}}{2\sqrt{2}+2} = 2+\sqrt{2} - 1}$$

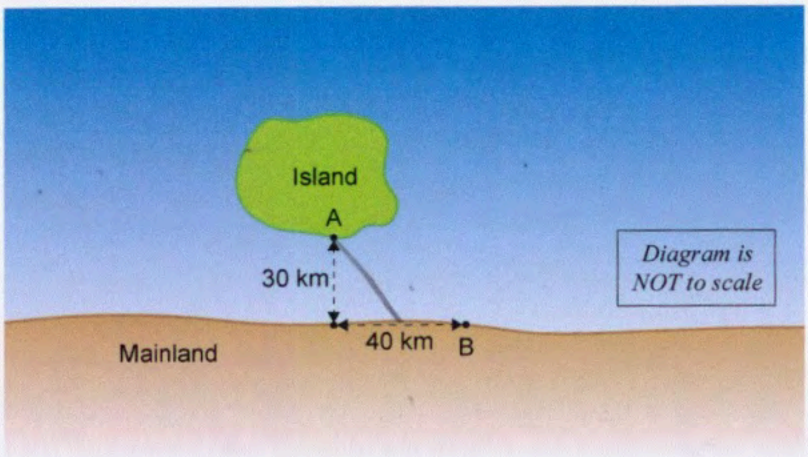
$$\cancel{u + (\sqrt{2}+1) -}\ \ \cancel{u + u^2(\sqrt{2}+1) - u + (\sqrt{2}+1) = \sqrt{2} + u(2+\sqrt{2})}$$

$$\cancel{(\sqrt{2}+1)u^2 - u + 1 = 0}$$

$$\Rightarrow u = \frac{2+\sqrt{2} \pm \sqrt{(6+4\sqrt{2}) - 4\sqrt{2} - 4}}{2\sqrt{2}+2} = 1 \pm \frac{\sqrt{2}}{2(\sqrt{2}+1)}$$

Extra space

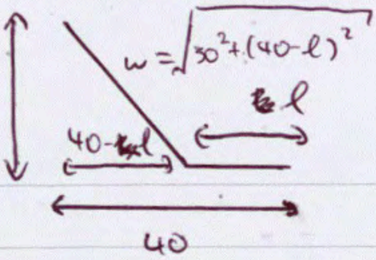
(c) A local council wishes to supply internet to an island 30 km from the mainland. One option involves running a cable between point A on the island and point B on the mainland, 40 km further down the coast, as shown below.



The cost to run a submarine cable under the water is $\$W$ per km, and the cost to run an underground cable on the mainland is $\$L$ per km, where $W > L$.

Use calculus to determine how the cable should be installed to minimise the overall cost of this option for two distinct scenarios:

- $W = 1.5L$ ①
- $W = 1.2L$ ②



①:

Cost $C(l) = \cancel{\ldots}^{\,l}\ 1.5(\cancel{900 + 1600}^{\,2500} = 80l + l^2)^{\frac{1}{2}}$

$C'(l) = 1.5 \cdot (2l - 80) \cdot \frac{1}{2}(2500 - 80l + l^2)^{-\frac{1}{2}} + 1$

Which has root at $l = \cancel{40}$ 13.2 (km) which (by GDC) is a local minimum

②: $C(l) = 1.2(2500 - 80l + l^2)^{\frac{1}{2}} + l$

Which ~~has local minimum at~~ is minimised at $l = 0$ for $l \geq 0$

$(uv)' = u'v + v'u$

QUESTION FOUR

$\Rightarrow \int u'v = uv - \int v'u$

(a) The graph below shows $y = \sin^{-1}(x)$, the inverse sine function, on its domain $-1 \le x \le 1$.

$\sin(u) = x$

Let

$\Rightarrow \frac{dx}{du} = \cos(u)$

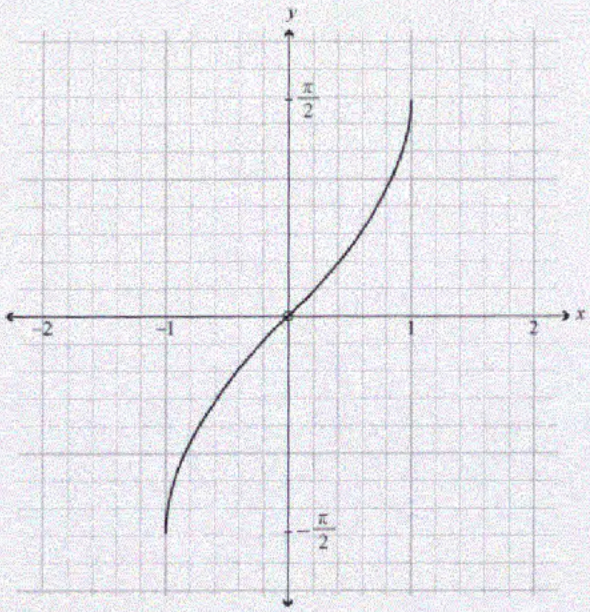
hence $\int \sin^{-1}(x)\,dx$

$= \int u \cos(u)\,du$

$v \quad u'$

$= u\sin(u) - \int \sin(u)\,du$

$= u\sin(u) - \cos(u)$



(i) Evaluate the definite integral:

$\sin(u) + u\cos(u) - \sin(u)$

$\int_0^1 \sin^{-1}(x)\,dx$

By GDC, $\int_0^1 \sin^{-1}(x)\,dx = 0.570796\ldots$

By hand, let $x = \sin(u) \Rightarrow dx = \cos(u)\,du$

$\Rightarrow \int \sin^{-1}(x)\,dx = \int u\cos(u)\,du = u\sin(u) - \int \sin(u)\,du$

$= [u\sin(u) + \cos(u)]$

$\Rightarrow \int_0^1 \sin^{-1}(x)\,dx = [\sin^{-1}(x)\cdot x + \cos(\sin^{-1}(x))]_0^1$

$= 1.570796\ldots - 1 = \boxed{0.570796\ldots}$

(ii) If $x = \sin y$, show that:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{1-x^2}}$$

Hence, or otherwise, find the minimum value of $g(x) = \sin^{-1}(2x) + \sin^{-1}\left(\frac{\pi}{4} - 2x\right)$.

You may assume that your answer is indeed a minimum.

$$\frac{dx}{dy} = \cos(y) \Rightarrow \frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{\cos^2(y)}} = \frac{1}{\sqrt{1-\sin^2(x)}} = \frac{1}{\sqrt{1-x^2}}$$

~~Substitute $x = \sin(y)$ into $g(x)$:~~

This implies $y = \sin^{-1}(x)$ $\therefore \frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$

~~$g(y) = \sin^{-1}(2\sin(y)) + \sin^{-1}(\frac{\pi}{4} - 2\sin(y))$~~

$$\text{Hence } g'(x) = \frac{2}{\sqrt{1-4x^2}} - \frac{2}{\sqrt{1-(\frac{\pi}{4}-2x)^2}} = 0$$

$$\Rightarrow 1-4x^2 = 1-\left(\frac{\pi}{4}-2x\right)^2 = 1-\left(\frac{\pi^2}{16} - \pi x + 4x^2\right)$$

$$\cancel{1}-4x^2 = \left(\cancel{1}-\frac{\pi^2}{16}\right) + \pi x - 4x^2$$

$$\Rightarrow \pi x - \frac{\pi^2}{16} = 0 \quad \therefore \boxed{x = \frac{\pi}{16}}$$

(b) Consider the function $f(x) = \left(\frac{1}{r^r}\right)^{\frac{1}{r+1}} \cdot x^r$, where r is a positive, real constant, and $x > 0$.

Find the value of r for which $f(x)$ is a solution of the differential equation $f'(x) = f^{-1}(x)$.

Note: if f and f^{-1} are inverse functions, it can be shown that $f\left(f^{-1}(x)\right) = x$ for all x in the domain of f^{-1}.

Let $\left(\frac{1}{r^r}\right)^{\frac{1}{r+1}} = k^{*} \Rightarrow f(x) = kx^r$

$f^{-1}(x)$ is reflection across $y=x \Rightarrow$ sub x for y:

$x = ky^r \Rightarrow y = \left(\frac{x}{k}\right)^{\frac{1}{r}} = \cancel{r^{r+1}x} \; r^{\frac{1}{r+1}} x^{\frac{1}{r}}$

~~Hence $f'(x) = \left(\frac{1}{r^{r^2+r}}\right) \cdot r x^{r-1}$~~

~~$= r^{1-r^2-r} x^{r-1} = r^{r+1} x^{1/r}$~~

Hence $f'(x) = r^{-\frac{r}{r+1}} \cdot r x^{r-1} = r^{1-\frac{r}{r+1}} x^{r-1} = r^{\frac{1}{r+1}} x^{\frac{1}{r}}$

$* \; 1 - \frac{r}{r+1} = \frac{r+1-r}{r+1} = \frac{1}{r+1}$

Hence $r^{\frac{1}{r+1}} x^{r-1} = r^{\frac{1}{r+1}} x^{\frac{1}{r}}$

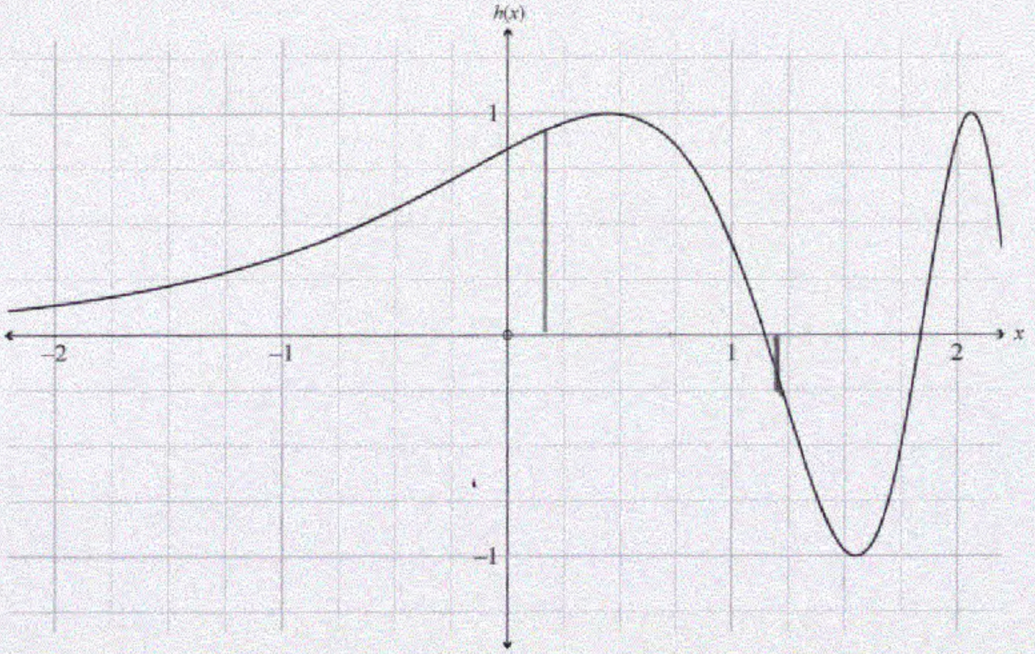
$\Rightarrow r-1 = \frac{1}{r} \Rightarrow r^2 - r - 1 = 0$

$\therefore r = 1.618, -0.618$

$* \; k = \left(\frac{1}{r^r}\right)^{\frac{1}{r+1}} = \cancel{\frac{1}{r^{r^2+r}}} = \frac{1}{r^{\frac{r}{r+1}}} = r^{-\frac{r}{r+1}}$

$\int u'v = uv - \int v'u$

(c) The graph below shows part of the graph of $h(x) = \sin(e^x)$.



(i) Find the value of $\lim_{x\to-\infty} h(x)$ and $\lim_{x\to\infty} h(x)$, or state clearly if any do not exist.

Support your answer with mathematical reasoning.

$\boxed{\lim_{x\to-\infty} h(x) = 0}$, since $\lim_{x\to-\infty} e^x = 0$ and $\lim_{x\to 0} \sin(x) = 0$

$\lim_{x\to\infty} e^x = \infty$. $\lim_{x\to\infty} \sin(x)$ does not exist, since $\sin(x)$ keeps on oscillating as $x\to\infty$ and does not converge to a value

$\therefore \lim_{x\to\infty} \sin(e^x)$ does not exist.

$\cos(u) - \cos(u)$

(ii) Find the exact value of a that generates the maximum value of the definite integral:

$I = \int_a^{a+1} \sin(e^x)\,dx$ Let $e^x = u \Rightarrow \frac{du}{dx} = e^x = u$

You may use the graph above to justify your answer.

Consider $\int \sin(e^x)\,dx$ with the substitution $u = e^x$

$\Rightarrow \frac{dx}{du} = e^x = u \Rightarrow \int \sin(e^x)\,dx = \int u\sin(u)\,du$

$= -u\cos(u) + \int \cos(u)\,du = \sin(u) - u\cos(u) = \sin(e^x) - e^x\cos(e^x)$

Hence $I = \int_a^{a+1} \sin(e^x)\,dx = \left[\sin(e^x) - e^x\cos(e^x)\right]_a^{a+1}$

$I(a) = \sin(e^{a+1}) - e^{a+1}\cos(e^{a+1}) - \sin(e^a) - e^a\cos(e^a)$

By GDC, $I(a)$ has a minimum at $a \approx 0.2075$

This is reasonable with the shape of $h(x)$: we would expect the max area to be somewhere around $0 \leq x \leq 1.25$, since the largest 'peak'/'bump' is there

Extra space if required.
Write the question number(s) if applicable.

QUESTION NUMBER

(b)

(3)(ii)

$\cancel{u} + u^2(\sqrt{2}+1) \cancel{-u} + (\sqrt{2}+1) = \sqrt{2} + u(2+\sqrt{2})$

$u^2(\sqrt{2}+1) - u(2+\sqrt{2}) + 1 = 0$

$\Rightarrow u = \dfrac{2+\sqrt{2} \pm \sqrt{6+2\sqrt{2}-4\sqrt{2}-4}}{2+2\sqrt{2}}$

$= \dfrac{2+\sqrt{2} \pm \sqrt{2}}{2+2\sqrt{2}} = \dfrac{1}{1+\sqrt{2}}$ OR $1 = \tan\left(\frac{x}{2}\right)$

$\Longrightarrow \frac{x}{2} = \frac{\pi}{4}, \frac{\pi}{8} \Rightarrow \boxed{x = \frac{\pi}{2}, \frac{\pi}{4}}$

$\Longrightarrow \boxed{y = \frac{\pi}{8}, \frac{\pi}{4}}$

Scholarship

Subject: Calculus

Standard: 93202

Total score: 24

Q	Score	Marker commentary
1	6	The candidate showed ability in changing the radical roots to fractional indices and therefore forming a cubic equation in Q1(a). In Q1(b), they successfully formed simultaneous equations by comparing the real and imaginary parts of both sides of the equation and then solved for the real roots. They demonstrated excellent understanding of the Fundamental Theorem of Algebra by transforming the original equation into a single-variable polynomial of degree 2026. Unfortunately, they forgot that 0 is also a root.
2	6	The candidate showed proficiency in finding the second derivative of a parametric function in Q2(a). They made a minor error in forming the equation using the given condition but were consistent in their work afterwards. In Q2(b), they showed ability in dealing with odd functions and, by examining patterns in small cases, were able to generalise and find the final answer. In 3c, they began by substituting the three points into the function and would have solved the problem had they also applied the given condition that the x-coordinates of D, E, and F sum to 25.
3	6	The candidate displayed fluency in applying trigonometric identities throughout this question. They successfully found the exact value of the definite integral in Q3(a) and correctly applied the compound angle formula in Q3(b)(i). However, their solution in Q3(b)(ii) was incomplete as they did not give the general solution of the trig equations. They were able to optimise a real-life problem by forming and differentiating an equation, but did not complete the work or provide a final answer.
4	6	The candidate demonstrated aptitude in integrating inverse trigonometric functions using substitution and then by parts in Q4(a). They also displayed skill in manipulating complex indices in Q4(b). By using the hint provided, they were able to form a quadratic equation by equating the indices of the variable x. Overall, they showed strong problem-solving strategies and attention to detail in handling both calculus and algebraic techniques.